Transit-time instability in Hall thrusters

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Longitudinal waves characterized by a phase velocity of the order of the velocity of ions have been recurrently observed in Hall thruster experiments and simulations. The origin of this so-called ion transit-time instability is investigated with a simple one-dimensional fluid model of a Hall thruster discharge in which cold ions are accelerated between two electrodes within a quasineutral plasma. A short-wave asymptotics applied to linearized equations shows that plasma perturbations in such a device consist of quasineutral ion acoustic waves superimposed on a background standing wave generated by discharge current oscillations. Under adequate circumstances and, in particular, at high ionization levels, acoustic waves are amplified as they propagate, inducing strong perturbation of the ion density and velocity. Responding to the subsequent perturbation of the column resistivity, the discharge current generates a standing wave, the reflection of which sustains the generation of acoustic waves at the inlet boundary. A calculation of the frequency and growth rate of this resonance mechanism for a supersonic ion flow is proposed, which illustrates the influence of the ionization degree on their onset and the approximate scaling of the frequency with the ion transit time. Consistent with experimental reports, the traveling wave can be observed on plasma density and velocity perturbations, while the plasma potential ostensibly oscillates in phase along the discharge. © 2005 American Institute of Physics. [DOI: 10.1063/1.1947796]

I. INTRODUCTION

Hall thrusters are plasma accelerators with crossed electric and magnetic fields primarily dedicated to satellite positioning and deep-space probe propulsion. Their operation is characterized by a wide spectrum of plasma instabilities, ranging from a few tens of kilohertz for the longitudinal breathing mode to a few megahertz for some azimuthal modes. The so-called transit-time instability has been experimentally identified as a longitudinal mode involving ion dynamics, with a frequency in the 100-500-kHz band. Interestingly, an ion-transit instability has also been repeatedly reported in numerical simulations.

Although a candidate explanation of the transit-time mechanism was first proposed by Esipchuk in his generalization of Morozov’s earlier stability study, some issues with its mathematical treatment have been pointed out (cf. Appendix). Numerical simulations of transit-time oscillations have more recently highlighted a splitting of the ion jet into several populations, likely related to the wave-breaking phenomenon. It must be noted, however, that wave breaking is a nonlinear process and thus has no bearing on the threshold of the instability.

By analogy with the one-dimensional dynamics of stellar winds, the present work proposes to mitigate the complexity of the wave-propagation problem in a Hall thruster by considering high-frequency/short-wave modes, the behaviors of which are conveniently analyzed within the frame of the Wentzel–Kramers–Brillouin (WKB) approximation. The governing equations and resulting perturbation equations are derived in Sec. II. In Sec. III, the WKB formalism is extended to account for an external source of forced oscillations, thus allowing the local behavior of acoustic waves to be analyzed separately from the perturbation associated to total current oscillations. The high-frequency approximation hence derived is coupled in Sec. IV with the equation of the discharge current for the idealized case of a fully supersonic ion flow, raising asymptotic formulas for the growth rates and eigenfrequencies of the instability.

II. GOVERNING EQUATIONS

A. Time-dependent equations

The governing equations are reduced to the one-dimensional continuity and momentum conservation for ions in a quasineutral plasma, with production of cold ions,

$$\frac{\partial n}{\partial t} + \frac{\partial nV}{\partial z} = \nu_p n,$$  \hspace{1cm} (1)

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial V^2}{\partial z} = \frac{eE}{M} - \nu_i V,$$  \hspace{1cm} (2)

together with a simplified Ohm’s law,

$$\frac{eE}{M} = -\frac{1}{2} \frac{\partial V^2}{\partial z} = \nu_i V.$$  \hspace{1cm} (3)

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The stationary solutions for simplicity, and electron–wall collisions were neglected for the sake of a priori. The recombination of ions at the walls and electron–wall collisions were neglected for the sake of simplicity.

From the quasineutrality hypothesis, it follows that the total current density
\[ J = nV - nU \] is independent of \( z \) [for convenience, \( J \approx I/(eS) \) is expressed in units of a particle flux density]. The time evolution of \( J \) is intrinsically determined by imposing a fixed potential drop between the inlet \((z=0)\) and the outlet \((z=L)\) of the flow domain, i.e.,
\[ \int_0^L E \, dz = \text{const.} \]

### B. Perturbation equations

Let us define the ion sound velocity and an effective electron-collision frequency as
\[ C = \sqrt{\frac{kT_e}{M}}, \]
\[ \nu_e = \frac{e}{M \mu_e}. \]

Substituting the electric field of Eq. (3) into Eq. (2), the ion dynamics equations become
\[ \frac{\partial n}{\partial t} + \frac{\partial nV}{\partial z} = \nu_i n, \]
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial V^2}{\partial z} + \frac{C^2 \partial n}{n \partial z} = \nu_e - \left( \nu_i + \nu_e \right) V. \]

Assuming small perturbations, the time-dependent variables are rewritten
\[ n(t,z) = \bar{n}(z) + \hat{n}(z) \exp(j\omega t), \]
\[ V(t,z) = \bar{V}(z) + \hat{V}(z) \exp(j\omega t), \]
\[ J(t) = \bar{J} + \hat{J} \exp(j\omega t). \]

The stationary solutions for \( \bar{n} \) and \( \bar{V} \) are given by
\[ \frac{d\bar{n}}{dz} = \nu_i - \frac{1 - \eta}{C^2 - \bar{V}^2} \]
with \( \eta \) defined as
\[ \eta = \frac{\bar{n} \bar{V}}{\bar{n}} \left( 1 + 2 \frac{\nu_i}{\nu_e} \right). \]

The perturbation equations are
\[ \left( j\omega - \nu_i + \frac{d\bar{V}}{dz} \right) \hat{n} + \bar{V} \frac{\partial \hat{n}}{\partial z} + \frac{\partial \bar{n}}{\partial z} \hat{V} + \bar{n} \frac{d\hat{V}}{dz} = 0, \]
\[ \frac{\bar{V}}{\bar{n}} \left( \frac{\partial \bar{V}}{\partial z} + \nu_i + \nu_e \right) \hat{n} + \frac{C^2 \partial \bar{n}}{\bar{n} \partial z} + j \omega \left( \frac{\partial \bar{V}}{\partial z} + \nu_i + \nu_e \right) \bar{V} \]
\[ + \frac{\bar{V}}{\bar{n}} \frac{d\bar{V}}{dz} = \frac{\hat{j}}{\bar{n}}. \]

It can be directly inferred from Eqs. (13) and (14) that if the solutions are regular for \( \bar{V} = C \), the sonic point is then crossed with \( \eta = 1 \). This is analogous to the situation of the Laval nozzle, where the sonic point is imperatively crossed at the throat.

### III. AMPLIFICATION OF ACOUSTIC MODES

#### A. Main assumptions

The system Eqs. (8) and (9) are similar to that of a one-dimensional isothermal flow in classical gas dynamics. In such media, the amplification or decay of propagating acoustic waves occur on a spatial scale comparable with that of the characteristic scale of plasma gradients. The evolution of the amplitude of ion-acoustic waves can be easily analyzed within the short-wave WKB approximation, which in essence requires that the gradient of any macroscopic property \( p \) be small enough with respect to the wavelength,
\[ \left| \nabla p \right|^{-1} \gg \text{wavelength}. \]

Taking into consideration that the phase velocity of acoustic waves is finite and that the characteristic gradients are scaled by \( \nu_i \) and \( \nu_e \) [see for instance Eqs. (13) and (14)], this assumption can be translated into the requirement that \( \omega \) should be larger than the characteristic frequencies of the system,
\[ \omega \gg \nu_i, \quad \omega \gg \nu_e. \]

It should be noted that the WKB approach cannot be expected to be strictly valid for the fundamental transit-time mode, because the wavelength of this mode is comparable with the length of the channel, and hence with the characteristic length of axial gradients. Therefore, only the behavior of higher-order modes shall be investigated.
B. High-frequency solution

In the high-frequency limit, the solution of Eqs. (16) and (17) can be split as

\[ \hat{n} = \hat{n}^{(0)} + \hat{n}^{(1)}, \]
\[ \hat{V} = \hat{V}^{(0)} + \hat{V}^{(1)}, \]

where the following are distinguished:

(i) a spatially nonoscillatory part (i.e., slowly varying in space), expressible as a series of inverse powers of \( \omega \),

\[ \hat{n}^{(0)}(z) = \sum_{k=1}^{\infty} \frac{1}{\omega^k} \hat{n}_k^{(0)}(z), \tag{22} \]
\[ \hat{V}^{(0)}(z) = \sum_{k=1}^{\infty} \frac{1}{\omega^k} \hat{V}_k^{(0)}(z), \tag{23} \]

(ii) a spatially oscillatory part, expressible as a WKB expansion in \( \omega \),

\[ \hat{n}^{(1)}(z) = \exp[j\omega \varphi(z)] \sum_{k=0}^{\infty} \frac{1}{\omega^k} \hat{n}_k^{(1)}(z), \tag{24} \]
\[ \hat{V}^{(1)}(z) = \exp[j\omega \varphi(z)] \sum_{k=0}^{\infty} \frac{1}{\omega^k} \hat{V}_k^{(1)}(z). \tag{25} \]

The nonoscillatory part is a special solution of Eqs. (16) and (17). It is easily verified that this solution is uniquely defined in terms of \( \hat{J} \) by equating terms of equal power of \( \omega \) in the perturbation equations, which up to the first-order in \( \omega^{-1} \) gives

\[ \hat{n}^{(0)} = \hat{J} \times O\left(\frac{1}{\omega^0}\right), \tag{26} \]
\[ \hat{V}^{(0)} = \hat{J} \times \left[-\frac{j}{\omega} \frac{\nu_e}{\hat{n}} + O\left(\frac{1}{\omega^0}\right)\right]. \tag{27} \]

It can be noted that the nonoscillatory part is proportional to the amplitude of total current oscillations, \( \hat{J} \), and is dominantly carried by \( \hat{V}^{(0)} \). This is related to the fact that the perturbations of the discharge current affects directly the velocity of ions through Eq. (9), as a result of the tight coupling between the electric field and the discharge current. For this reason, \( \hat{V}^{(0)} \) shall also be referred to as the feedback perturbation due to its ability to transmit total current variations through the flow. The actual value of \( \hat{J} \) will be determined in Sec. IV from the boundary condition on the potential drop.

The oscillatory part is the general solution of the homogeneous problem associated with Eqs. (16) and (17). One may think of the oscillatory part as a wave, the spatial guiding center of which is the nonoscillatory part. Its solution is more conveniently expressed in terms of the acoustic eigenmodes,

\[ \hat{J}_\pm = \frac{\hat{J}^{(1)}}{\sqrt{C \pm \sqrt{C^2 - \beta \hat{n}}} / \hat{n}}, \tag{28} \]

which correspond to the Riemann invariants. The first-order WKB approximation of the oscillatory part is then given as

\[ \hat{J}_\pm \approx const \times \exp\left[ \int \frac{g_s - j\omega}{\hat{V} \pm C} dz + O\left(\frac{1}{\omega}\right) \right], \tag{29} \]

where the WKB growth rate of the eigenmodes is

\[ g_s = -\frac{\nu_e + 2
\nu}{2} \left[ \frac{2 \pm M}{M \pm 1} \frac{1}{\eta_i - 1} + 1 \pm M \right], \tag{30} \]

with \( M \) the (time-averaged) Mach number,

\[ M = \frac{\hat{V}}{C}. \tag{31} \]

It is readily seen that \( \eta_i \) reduces to the ion current fraction when the local ionization frequency vanishes. It comes out from Eq. (29) that \( \hat{J}_+^{(1)} \) and \( \hat{J}_-^{(1)} \) are the spatial profiles of acoustic waves with phase velocities, respectively, tending to \( \hat{V} + C \) and \( \hat{V} - C \) at high frequency. If \( \omega \) is purely real, e.g., if sinusoidal oscillations are forced at some location \( z_0 \), then the sign of \( g_s \) indicates whether the waves \( \hat{J}_+^{(1)} \) are locally amplified \( (g_s > 0) \) or damped \( (g_s < 0) \) as they propagate away.

From their respective definitions, \( \eta_i \) and \( \beta \) both carry the sign of \( \hat{V} \) so that the study of the behavior of \( \hat{J}_+^{(1)} \) is limited to the quarter planes,

\[ \eta_i M \geq 0. \tag{32} \]

Assuming additionally that the Bohm condition at the anode sheath sets the lower bound \( M > -1 \), it can be verified from Eq. (30) that

\[ g_s < 0, \tag{33} \]

i.e., \( \hat{J}_+^{(1)} \) is unconditionally damped as it propagates away from a point of forced oscillations. The behavior of \( \hat{J}_-^{(1)} \), on the other hand, is characterized by distinct regions of growth and decay, identified in Fig. 1, split by the separatrix

\[ \eta_i = M(2 - \beta M). \tag{34} \]

It must be pointed out, however, that a more rigorous formulation of the hypothesis (19) would exhibit a well-known deficiency of the WKB approximation for \( \hat{J}_-^{(1)} \) in the vicinity of the sonic points \( (M = \pm 1) \). For the sake of simplicity, our investigation of the transit-time instability will thus be restricted to the case of a fully supersonic ion flow.

IV. STABILITY OF A SUPersonic Ion FLOW

A. Assumptions and boundary conditions

Since acoustic waves eventually leave the interelectrode domain, their amplification does not, by itself, warrant the appearance of an instability. A feedback mechanism, provided here by the total current \( \hat{J} \), is yet necessary to lead to self-sustained oscillations.
Valuable insights about the transit-time instability can be gathered from a study of the simplified case, where a fully supersonic ion flow is accelerated between two electrodes, assuming constant electron mobility, constant temperature, and no local ionization,

\[ \nu_e = \text{const}, \]

\[ C = \text{const}, \]

\[ \nu_i = 0. \]

Hypothesis (37) implies also that the time-averaged ion current fraction is constant over the domain, hence

\[ \eta = \frac{\bar{n} V}{J} = \text{const}. \]

Since the potential of each electrode is fixed, the perturbation of plasma potential vanishes at both boundaries,

\[ \Phi|_{z=0} = \Phi|_{z=L} = 0. \]

Besides, the density and the velocity at the inlet are imposed, meaning that their total perturbations vanish at \( z = 0 \),

\[ \hat{n}^{(1)}|_{z=0} + \hat{n}^{(1)}|_{z=0} = 0, \]

\[ \hat{V}^{(0)}|_{z=0} + \hat{V}^{(0)}|_{z=0} = 0. \]

Remembering that \( \hat{n}^{(0)} \) and \( \hat{V}^{(0)} \) are given by Eqs. (26) and (27) independently of boundary conditions, we can directly express \( \hat{n}^{(1)} \) and \( \hat{V}^{(1)} \) at \( z = 0 \). Then from Eq. (28) it can be found that both eigenmodes present an identical asymptotic development at that boundary, up to the first order in \( \omega^{-1} \),

\[ \hat{\lambda}^{(1)}_{k} = M \times \left[ \frac{j}{\omega \sqrt{CNl_{\infty}} - 1} + \mathcal{O} \left( \frac{1}{\omega^2} \right) \right]. \]

The fact that \( \hat{\lambda}^{(1)}_{k} \) and \( \hat{\lambda}^{(1)}_{k} \) are proportional to \( \hat{J} \) at \( z = 0 \) shows that despite \( n \) and \( V \) being unperturbed at that point, wave trains are triggered as soon as the total current is perturbed.

**B. Resolution**

We shall now close the system of equations using the boundary condition on the electric potential. From Eqs. (3) and (4), the plasma potential with respect to the anode \( (z = 0) \) is

\[ \Phi = \frac{1}{\mu_e} \int_{0}^{z} \left( V - \frac{J}{n} \right) dz + \frac{kT_e}{e} \int_{0}^{z} \frac{1}{n} dn dz, \]

which leads to the perturbed plasma potential,

\[ \Phi = \frac{M \nu_e}{e} \left[ \int_{0}^{z} \left( \frac{\bar{n} \hat{V}^{(1)} + \hat{V}^{(1)}}{\bar{n}} \right) dz - \int_{0}^{z} \frac{1}{n} dz \right] + \frac{MC^2}{e} \]

\[ \times \left( \frac{n}{n} - \frac{n}{n} \right)_{z=0}. \]

Using Eqs. (20), (21), (26), and (27), the former relationship can be written in the form

\[ \Phi = \frac{M \nu_e}{e} \left[ \int_{0}^{z} \left( \frac{\bar{n} \hat{V}^{(1)} + \hat{V}^{(1)}}{\bar{n}} \right) dz - \int_{0}^{z} \frac{1}{n} dz \right] + \frac{MC^2}{e} \]

\[ \times \left( \frac{n}{n} - \frac{n}{n} \right)_{z=0} \cdot \hat{J} \times \mathcal{O} \left( \frac{1}{\omega} \right). \]

Let us now examine the relative orders of magnitude of the quantities numbered 1, 2, 3, and 4 in Eq. (45). Since the complex phases of \( \hat{n}^{(1)} \) and \( \hat{V}^{(1)} \) oscillate rapidly and proportionally to \( \omega \), integral 1 is \( \mathcal{O}(\omega^{-1}) \) with respect to quantity 3 and can be accordingly neglected. Additionally, Eqs. (26) and (40) imply that \( \hat{n}^{(1)} \) at \( z = 0 \) is of the order \( \hat{J} \times \mathcal{O}(\omega^{-2}) \), which allows, in turn, quantity 4 to be neglected. Using Eq. (28) to express quantity 3, a simplified expression of the permeation potential is finally obtained as

\[ \Phi = -\hat{J} \frac{M \nu_e}{e} \int_{0}^{z} \frac{1}{n} dz + \frac{MC^2}{e} \left( \hat{\lambda}^{(1)}_{k} - \hat{\lambda}^{(1)}_{k} \right) + \hat{J} \times \mathcal{O} \left( \frac{1}{\omega} \right). \]

The eigenmodes \( \hat{\lambda}^{(1)}_{k} \) and \( \hat{\lambda}^{(1)}_{k} \) can be subsequently substituted by their expressions from Eq. (29), where the constant factors are given by Eq. (42). Expressing \( \Phi \) at \( z = L \) and applying Eq. (39), an asymptotic form of the eigenfrequency equation for transit-time modes is obtained,

\[ \frac{C}{2j \omega \bar{n}_{\infty}} \left( \exp \int_{0}^{L} \frac{g_j - j \omega}{V + C} dz - \exp \int_{0}^{L} \frac{g_j - j \omega}{V - C} dz \right) \]

\[ = \int_{0}^{1} \frac{1}{n} dz + \mathcal{O} \left( \frac{1}{\omega} \right). \]
C. Approximate frequency and growth rate

As we are mainly interested in the transition from stable to unstable operation [\(\text{Im}(\omega) = 0\)], it is sufficient to look for approximate solutions of Eq. (47) for which the growth rate of the instability, \(\text{Im}(\omega)\), is greater than the lower bound of \(g_\tau(z)\). Indeed, Eq. (33) ensures that \(g_\tau(z) < 0\). Under this hypothesis, the complex amplitude of the first exponential function in Eq. (47) decays and remains of the order \(\mathcal{O}(1)\). Consequently, the second exponential function must be of the order \(\mathcal{O}(\omega)\) to balance the magnitude of the right-hand side of Eq. (47). Ignoring thus the first exponential function, one can easily relate the complex phases of both sides of Eq. (47) to obtain the asymptotic behavior of the frequency,

\[
\text{Re}(\omega) \approx \frac{m + \frac{3}{4}}{\tau_\omega}, \quad m \in \mathbb{N},
\]

where \(m\) denotes the mode number and \(\tau_\omega\) the transit time of the \(\lambda_\omega\) wave,

\[
\tau_\omega = \int_0^L \frac{dz}{V - C}.
\]

Note that, since the flow is assumed to be supersonic, the condition \(V > C\) is always fulfilled in Eq. (49). Finally, by relating the complex moduli of both sides of Eq. (47) the growth rate of the instability is obtained as

\[
-\text{Im}(\omega) \approx \frac{1}{\tau_\omega} \times \left[ -\ln \left( \frac{4m + 3}{C \tau_\omega} \int_0^L \frac{n^i dz}{\bar{n}} + \int_0^L g dz \right) \right].
\]

D. Case study and discussion

For illustration purposes, we shall set the inlet Mach number and the potential drop to

\[
\mathcal{M}_{i=0} = 2, \quad \int_0^L E dz = 6 \frac{MC^2}{e}.
\]

With these assumptions, the outlet Mach number is

\[
\mathcal{M}_{i=L} = 4,
\]

which is a realistic value for a Hall thruster.

The corresponding approximate growth rates derived from Eq. (50) are plotted in Fig. 2 as functions of the ion current fraction. Note that only the modes with \(m \geq 5\) are reported, as the high-frequency approximation significantly departs from the exact solution below this threshold. The prediction that low-\(m\) modes are the most unstable can, nevertheless, be extrapolated down to the lowest modes, as confirmed by numerical solutions of the exact perturbation equations (16), (17), and (44), which show that the fundamental mode is indeed the least stable, with a threshold at \(\eta_i = 0.79\) (to be compared with \(\eta_i = 0.84\) for \(m = 5\)).

Figure 2 clearly emphasizes the relationship between the onset of instabilities and the ion current fraction \(\eta_i\). This behavior could, of course, be expected since, on the one hand, \(\eta_i\) governs the WKB amplification of the \(\dot{\la}_i^{(1)}\) wave (see Fig. 1) and, on the other hand, the overall spatial amplification of this wave,

\[
\int_0^L \frac{g dz}{V - C},
\]

directly contributes to the growth rate of the instability, as can be seen from Eq. (50).

Using now as an example the propagation of the marginally stable \(m = 5\) mode [\(\text{Im}(\omega) = 0\), \(\text{Re}(\omega) = 7.1 \times v_i\), and \(\eta_i \approx 0.84\)] displayed in Figs. 3 and 4, the underlying mechanism of the transit-time instability can be depicted as follows:

(i) The total current perturbation \(\dot{J} \exp(\imath \omega t)\) is transmitted through the plasma via the feedback perturbation \(\dot{V}^{(0)} \exp(\imath \omega t)\), which, according to Eq. (27), is characterized by a standing-wave pattern shifted by \(\pi/2\) with respect to the total current perturbation [Fig. 3(a)].

(ii) Since the total velocity perturbation \(\dot{V}^{(0)} + \dot{V}^{(1)}\) vanishes at the inlet, the feedback transmitted by \(\dot{V}^{(0)}\) toward the boundary \(z = 0\) is reflected by \(\dot{V}^{(1)}\),

\[
\dot{V}^{(1)} \exp(\imath \omega t) = -\dot{V}^{(0)} \exp(\imath \omega t) \quad \text{at} \quad z = 0.
\]

(iii) The perturbation of \(\dot{V}^{(1)}\) at the boundary \(z = 0\) is propagated downstream (and in the case at hand, amplified) by the acoustic wave, the phase velocity of which is \(V - C\) [Fig. 3(b)].
The perturbations of $\hat{n}^{(1)}$ and $\hat{v}^{(1)}$ generated by the wave propagation [Figs. 3(b) and 4] affect the global resistivity of the plasma, which, in turn, generates a new perturbation of the total current aimed at maintaining a constant potential drop.

Although it is not possible to draw enough information from available experimental results to firmly confirm the suggested oscillation mechanism, it appears at least possible to resolve an apparent discrepancy between time-resolved measurements of the plasma potential distribution\textsuperscript{3,4} and later plasma density measurements.\textsuperscript{5,6} While the former studies concluded to the existence of a standing wave, the latter highlighted the presence of traveling waves. Indeed, Fig. 4 shows that the propagation of the sonic wave is not directly reflected by the plasma potential. This follows from the fact that the plasma potential does not characterize the local state of the plasma (as would the density or the electric field), but rather the whole plasma column up to the reference boundary. In practice, thus, the plasma potential oscillates almost in phase through most of the channel where the electric field is low, with only a slight phase shift confined to the high-electric-field region.
The saturation mechanism of this mode lies beyond the scope of the linear stability study, but the following nonlinear effects can be expected to play a significant role in this respect:

1. According to numerical simulations, transit-time oscillations seem more likely but at the same time more strongly saturated when the secondary electron emission of the discharge channel is high.\(^1\)
2. Quasineutral simulations suggest that transit-time oscillations can induce wave steepening and eventually, wave breaking.\(^1\) This can be thought of as a source of dissipation since it contributes to the spreading of the energy distribution of ions.

Some insights about the energy source of transit-time modes can be gained by applying the methodology outlined in Ref. 17. In particular, the excitation of acoustic waves can be shown to induce a systematic decrease of the mean ion velocity proportional to the wave energy. Oscillations are therefore not necessarily sustained by an increase of the average discharge current, but mostly result from a direct transfer of energy from the ion flow to the wave. Interestingly, by neglecting ionization it can be shown that the mean discharge current actually decreases under the effect of oscillations when the Mach number at the outlet is higher than 1.44.

As a final note, even though Eqs. (1)–(3) ignore all convective phenomena associated with electrons, one cannot in principle discard the existence of a similar instability related to electron transit. This seems rather unlikely in practice, however, since the information propagated by electrons is mainly encoded in their energy. In the strongly magnetized region, their short cyclotron radius and the relative intensity of inelastic collisions make the electron energy in a large part determined by the local balance between gains and losses, convective terms being small in comparison. Therefore, in contrast to ions, the information carried by electrons dissipate over small distances.

V. CONCLUSION

A high-frequency analysis of the transit-time instability for a supersonic ion flow reveals that it is triggered by a resonance of longitudinal acoustic waves, ensured by a quasi-instantaneous feedback from cathode to anode carried by discharge-current perturbations. The asymptotic behavior of its characteristic frequency for high mode numbers \(m\) was found to obey the relationship

\[
\omega = \frac{m + \frac{3}{2}}{\tau_c},
\]

showing thus an inverse dependence with respect to the transit time between the electrodes of the most amplified sonic wave. Since the phase velocity of ion acoustic waves, \(V_{\pm c}\), is of the order of the ion velocity in the acceleration region of Hall thrusters, the proposed mechanism seems adequate to qualitatively explain experimentally measured frequencies.\(^3\)

The onset of this instability was shown to involve the amplification of acoustic waves, the spatial growths of which are governed by the ion current fraction and by the intensity of the local ionization, synthesized in a single parameter,

\[
\eta = \frac{\bar{n}_i V}{f (1 + 2 \nu_i/\nu_e)}.
\]

In the acceleration region, one of the acoustic waves is strongly amplified when \(\eta\) approaches unity, which is consistent with experimental observations and simulations reporting that the transit-time instability is more likely to appear for high ionization levels.\(^3,11\)

It must be underlined, however, that such a study cannot pretend to provide more than qualitative insights. Even though the general transit-time mechanism outlined is expected to be valid for all modes, quantitative predictions for the fundamental transit-time mode are, in nature, very unreliable, owing to its low frequency and long wavelength. In practice, only the higher transit-time modes \((m \geq 5)\) can be considered as reasonably approximated by the asymptotic solution. Additionally, the transonic character of the ion flow in an actual Hall thruster makes the problem much more intricate than in the idealized supersonic flow investigated. Finally, even though the random energy of the ion beam in Hall thrusters is typically a few times smaller than the electron temperature, the neglect of Landau damping cannot be rigorously justified for high-frequency modes.

The nonlinear evolution of acoustic waves also deserves additional investigations. Quasineutral numerical models of Hall thruster plasmas using a kinetic description of ions predict that ions could eventually split into a slow and a fast beam,\(^11\) a situation that can be compared to the multivalued solutions of fluid equations arising at the point of wave breaking. Although wave breaking is discarded in most plasmas on the ground that electrostatic forces (double layers) eventually prevent the steepening process, a definite statement on this question would require the quasineutrality hypothesis to be priorly lifted.

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APPENDIX: CONSISTENCY OF LOCAL STABILITY CRITERIA IN NONUNIFORM MEDIA

Former transit-time oscillations studies by Esipchuk and Tilinin\(^1\) as well as several other linear-stability studies for nonuniform plasmas appear to suffer from a mathematical deficiency in the way gradients are accounted for. In order to avoid undue complications, the somewhat simpler analysis of Morozov for azimuthal waves on the premises of which is built Esipchuck’s work will be used as a case in point. In the following, the original variable names have been preserved.\(^12\)
Starting from Morozov’s initial equation system,
\[ \frac{\partial \mathbf{n}}{\partial t} + \mathbf{H} \cdot \nabla \mathbf{n} = 0, \]
\[ M \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} \right) = - \frac{\partial \Phi}{\partial t}, \]
\[ M \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} \right) = - \frac{\partial \Phi}{\partial t}, \]
\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{H} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{H} = 0, \]
\[ \frac{\partial \mathbf{H}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{H} = 0. \]

Performing then the substitution
\[ \begin{pmatrix} \hat{n} \\ \vdots \\ \hat{u} \end{pmatrix} \rightarrow \exp(i \omega t + j l x + j m y), \]
and defining \( k^2 = l^2 + m^2 \), one gets a dispersion-like relationship,
\[ \left( \omega + ilv \right)^2 + \frac{k^2}{\mu_0} \left( \omega + mU \right) L_H = 0, \]
which, under the condition
\[ \frac{\partial}{\partial x} \left( \frac{\mathbf{H}}{n} \right) < 0 \]
and for \( k \) real, has solutions for \( \omega \) with \( \text{Im}(\omega) < 0 \), thus leading to instability.

This procedure seems to be the one carried out by Morozov in deriving the above criterion. In the spirit of this method, however, it should be possible to substitute the six original variables, \( n, v, U \), for their Fourier counterpart directly in the initial equation system, since it is absolutely identical from a mathematical point of view. Doing so, another dispersion-like relationship is obtained, the short-wave approximation of which reads
\[ \left( \omega + ilv \right)^2 + \frac{k^2}{\mu_0} \left( \omega + mU \right) L_n = 0, \]
where
\[ L_n = - \left( \frac{\partial n}{\partial x} \right)^{-1}. \]

This unexpectedly leads, however, to a different stability criterion,
\[ \frac{\partial n}{\partial x} > 0. \]

The latter result clearly points to a flaw in the method, as there exists no objective way to select which criterion is appropriate. Indeed, the spatial dependency of the wave number makes it elusive to seek for a dispersion relation that accounts for gradients. This can be highlighted for the following simple scalar perturbation equation,
\[ \frac{\partial u}{\partial t} + a(x) \frac{\partial u}{\partial x} = bu, \]
for which it can be rigorously established that \( \dot{u} \) decays along the direction of propagation (given by the sign of \( a \)) when \( b < 0 \). Using the substitution \( \dot{u} \rightarrow \dot{u} \exp(i \omega t + j l x) \), one obtains
\[ \omega = ak - jb, \]
which, in appearance, seems consistent with the fact that \( b < 0 \) leads to a decay. If one differentiates the perturbation equation prior to substitution, however, the following relationship is obtained:
\[ \omega = ak + j \left( \frac{\partial a}{\partial x} - b \right). \]
This wrongly suggests that, for an appropriate gradient of \( a \), waves may grow even if \( b < 0 \). It is worth reminding that differentiation was also used in Morozov’s derivation and indeed introduced an additional gradient \( \partial B_1 / \partial x \) in comparison with the case where substitution is performed in the primary equation system. It can be verified that the geometrical-optics (WKBJ) method does not suffer from such a pathology.
It can be noted that neither Morozov’s equation system nor the simplified problem considered above gives rise to an instability in the usual sense of that word. Just as in Sec. III, it can only be determined whether the waves generated by forced oscillations are amplified or decay as they propagate. In the case of transit-time oscillations, it is the interplay between such propagating waves and the discharge current that ultimately gives rise to an instability (Sec. IV).